Indian Statistical Institute
First Semester 2005-2006
Mid Semestral Exam
B.Math I Year
Analysis II

Time: 3 hrs Date:27-02-06 [Max marks :40]

1. Let $f_1, f_2, \ldots$ be a sequence of real valued Riemann integrable functions on $[a,b]$. Let $g : [a,b] \to R$ be a function such that $0 = \lim\limits_{n\to\infty} \sup\limits_{a\le x\le b} |f_n(x) - g(x)|$. Show that

   a) g is Riemann Integrable

   b) $\int\limits_a^b f_n(x)dx \to \int\limits_a^b g(x)dx$. [4]

2. Give an example of an interval $[a,b]$, real valued Riemann integrable function $f, f_1, f_2, \ldots$ on $[a,b]$ such that $f_n(x) \to f(x)$ as $n \to \infty$ for each x. But $\int\limits_a^b f_n(x)\ dx$ does not converge to $\int\limits_a^b f(x)\ dx$. Verify that $\sup\limits_{a\le x\le b} |f_n(x) - f(x)|$ does not converge to 0. [5]

3. a) For any function $f : I \to R$ with $|f(x)| \le M$ for all x in the interval I, show that

$$\left|\sup_{x\in I}[f(x)]^2 - \inf_{x\in I}[f(x)]^2\right| \le 2M \left|\sup_{x\in I} f(x) - \inf_{x\in I} f(x)\right|$$

   [2]

   (b) Let $g : [a,b] \to R$ be any bounded Riemann Integrable function. Show that g^2 is also Riemann integrable. [2]

   (c) Give an example of a Riemann integrable function $h : [0,1] \to R$ such that h^2 is not Riemann integrable and prove your claim. [1]

4. Let $f : [0,a] \to R$ be $(k+1)$ times differentiable and $f, f^{(1)}, \ldots f^{(k+1)}$ are all continuous. Prove the following Taylors expansion for f with remainder

$$f(a) = f(0) + a\frac{f^1(0)}{1!} + \frac{f^{(2)}(0)}{2!}a^2 + \frac{f^{(k)}(0)}{k!}a^k + \int\limits_0^a \frac{(a-t)^k}{k!} f^{(k+1)}(t)\ dt$$

   [4]

5. Let

$$f(x,y) = \frac{x^3}{x^2+y^2} \text{ for } (x,y) \neq (0,0)$$
$$f(0,0) = 0$$

a) Show that f is continuous on R^2 [2]

b) Calculate $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ on $R^2 \backslash (0,0)$ [1]

c) Show that $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are continuous on $R^2 \backslash (0,0)$

d) Show that f has total derivative at each point of $R^2 \backslash (0,0)$ [Hint: quote any theorem you know] [1]

e) Calculate directional derivative $(D_{\underset{\sim}{u}} f)(\underset{\sim}{\alpha})$ for each $\underset{\sim}{\alpha}$ in $R^2 \backslash (0,0)$ and each direction $\underset{\sim}{u}$ [Hint: Same as in d] [1]

f) Show that $D_{\underset{\sim}{u}} f(\underset{\sim}{0})$ exists for any direction $\underset{\sim}{u}$ [2]

g) Show that f has no total derivative at $\underset{\sim}{0}$ [2]

6. Let $S : M_{k\times k}(R) \to M_{k\times k}(R)$ by the function $S(A) = A^2$. Find the total derivative of S at each point of $M_{k\times k}(R)$ [3]

7. By Lagrange's method, show that a polygon of n sides with all the vertices on a circle has maximum perimeter when it is a regular polygon

[5]

8. Let $f : R^2 \longrightarrow R$ be given by $f(x,y) = x^4+x^2y+y^2$. Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y^2}$. Find all local minima or local maxima for f. Show that $(0,\ 0)$ is a local minima. [5]